Hidden nonlinear supersymmetry of finite-gap Lamé equation

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A bosonized nonlinear (polynomial) supersymmetry is revealed as a hidden symmetry of the finite-gap Lamé equation. This gives a natural explanation for peculiar properties of the periodic quantum system underlying diverse models and mechanisms in field theory, nonlinear wave physics, cosmology and condensed matter physics.

PACS numbers: 11.30.Pb; 11.30.Na; 03.65.Fd

Supersymmetry [1], as a fundamental symmetry providing a natural mechanism for unification of gravity with electromagnetic, strong and weak interactions, still waits for experimental confirmation. On the other hand, in nuclear physics supersymmetry was predicted theoretically [2] and has been confirmed experimentally [3] as a *dynamic* symmetry linking properties of some bosonic and fermionic nuclei. It would be interesting to look for some physical systems whose special properties could be explained by a hidden *ordinary* (not a dynamic) supersymmetry.

In the present Letter we show that the quantum system described by the finite-gap Lamé equation possesses a hidden supersymmetry. A very unusual nature of the revealed supersymmetry is that it manifests as a *nonlinear* symmetry of a *bosonic* system without fermion (spin) degrees of freedom. This means that we find here a kind of a *bosonized* supersymmetry giving a natural explanation for peculiar properties of the periodic quantum problem underlying many physical systems.

The Lamé equation first arose in solution of the Laplace equation by separation of variables in ellipsoidal coordinates [4], and one of its early applications was in the quantum Euler top problem [5]. It plays nowadays a prominent role in physics appearing in such diverse theories as crystals models in solid state physics [6, 7], exactly and quasi-exactly solvable quantum systems [8, 9], integrable systems and solitons [10, 11], supersymmetric quantum mechanics [12], BPS monopoles [13], instantons and sphalerons [14, 15], classical Ginzburg-Landau field theory [16], Josephson junctions [17], magnetostatic problems [18], inhomogeneous cosmologies [19], Kaluza-Klein theories [20], chaos [21], and preheating after inflation modern theories [22]. Most often, the Lamé equation appears in physics literature in the Jacobian form of a one-dimensional Schrodinger equation with a doubly periodic potential,

$$H_j\Psi = E\Psi, \quad H_j = -\frac{d^2}{dx^2} + j(j+1)k^2\,\mathrm{sn}^2(x,k), \quad (1)$$

where $\mathrm{sn}(x,k) \equiv \mathrm{sn}\,x$ is the Jacobi elliptic odd function with modulus k ($0 < k < 1$), and real and imaginary periods $4K$ and $2iK'$, $K = K(k)$ is a complete elliptic integral of the first kind, and $K' = K(k')$, $k'^2 = 1 - k^2$ [4, 23]. A remarkable property of this equation is that at integer values of the parameter $j = n$, its energy spectrum has exactly n gaps, which separate the $n+1$ allowed energy bands. The $2n+1$ eigenfunctions associated to the boundaries $E_i(n)$, $i = 0, 1, \ldots, 2n$, of the allowed energy bands $[E_0, E_1]$, $[E_2, E_3], \ldots, [E_{2n}, \infty]$ are given by polynomials ('Lamé polynomials') of degree n in the elliptic functions $\mathrm{sn}\,x$, $\mathrm{cn}\,x$ and $\mathrm{dn}\,x$. These polynomials have real periods $4K$ or $2K$, and the boundary energy levels $E_i(n)$ are *non-degenerated*. The states in the interior of allowed zones are described by the quasi-periodic Bloch-Floquet wave functions (which can be expressed in terms of theta functions [4]) of quasi-momentum $\kappa(E)$,

$$\Psi^{\pm}_E(x + 2K) = \exp(\pm i\kappa(E))\Psi^{\pm}_E(x).$$

Every such interior energy level is *doubly degenerated*. For any non-integer value of the parameter j, Eq. (1) has an *infinite* number of allowed and prohibited zones.

The double degeneration of the energy levels is typical for a quantum mechanical system with $N = 2$ supersymmetry. But the presence of $2n + 1$ edge-bands singlet states in the n-gap Lamé equation indicates on an unusual, nonlinear character [24, 25] of a possible hidden supersymmetry. To reveal it, one notes that in the limiting case $k = 1$ we have $K = \infty$, $K' = \frac{\pi}{2}$, and system (1) reduces to the Pöschl-Teller quantum system given by the potential

$$U(x) = -j(j+1)\,\mathrm{sech}^2 x + j(j+1).$$

The latter, as it was shown recently in [26], at $j = n$ possesses a hidden polynomial supersymmetry [24, 25] of order $2n+1$ generated by the supercharges Q_n and $\tilde{Q}_n = iRQ_n$,

$$[Q_n, H_n] = [\tilde{Q}_n, H_n] = 0, \quad \{Q_n, \tilde{Q}_n\} = 0, \quad (2)$$

$$Q_n^2 = \tilde{Q}_n^2 = P_{2n+1}(H_n), \quad (3)$$

where $P_{2n+1}(H_n)$ is some polynomial of the degree $2n+1$ of the Hamiltonian H_n, R is a reflection, $R\Psi(x) =$

$\Psi(-x)$, identified as the grading operator,

$$[R,H_n]=0,\quad \{R,Q_n\}=\{R,\tilde{Q}_n\}=0,\quad R^2=1,\quad (4)$$

and Q_n is a self-conjugate local differential operator of degree $2n+1$. Based on this observation, first we note that in the trivial case of the free particle system with $j=0$ characterized by the one allowed ('conduction') band $[E_0(0),\infty]$, $E_0(0)=0$, the odd first order differential operator $Q_0=-iD$, $D=\frac{d}{dx}$, is identified as the supercharge. For the one-gap Lamé system (1) with $j=1$, let us look for the self-conjugate integral of motion Q_1, $[Q_1,H_1]=0$, in the form of the third order differential operator. A direct check shows that

$$iQ_1=D^3+fD+\frac{1}{2}f',\qquad (5)$$

is the odd integral, $\{R,Q_1\}=0$. Here

$$f:=1+k^2-3k^2\,\mathrm{sn}^2x,\qquad (6)$$

$f'=Df$. The double-periodic elliptic function f with periods $2K$ and $2iK'$ satisfies the elliptic curve equation

$$(f')^2=\tfrac{4}{3}(a_1-f)(f-a_2)(f-a_3),\qquad (7)$$

whose characteristic roots are

$$a_1=f(0)=1+k^2,\quad a_2=f(K)=1-2k^2,\\ a_3=f(K+iK')=k^2-2,\qquad (8)$$

$a_1+a_2+a_3=0$. Differentiation of Eq. (7) gives the identities

$$f''+2f^2=2b^2,\quad D^l(D^2+2f)f=0,\qquad (9)$$

where $b^2=-\frac{1}{3}(a_1a_2+a_1a_3+a_2a_3)=k^4-k^2+1$, $l=1,2,\ldots$. Using these relations, one finds that $Q_1^2=P_3(H_1)$,

$$P_3(H_1)=(H_1-E_0(1))(H_1-E_1(1))(H_1-E_2(1)).$$

The energies

$$E_0(1)=k^2,\quad E_1(1)=1,\quad E_2(1)=1+k^2\qquad (10)$$

correspond here to the eigenfunctions $\Psi_0^{(1)}=\mathrm{dn}\,x$, $\Psi_1^{(1)}=\mathrm{cn}\,x$, $\Psi_2^{(1)}=\mathrm{sn}\,x$, which form a zero-mode subspace (kernel) of the supercharge Q_1. The states in the interior of the two allowed zones are described by the quasi-periodic eigenfunctions

$$\Psi_E^{\pm}=\frac{H(x\pm\alpha)}{\Theta(x)}\exp(\mp xZ(\alpha)),\qquad (11)$$

where $H(x)$, $\Theta(x)$ and $Z(x)$ are the Jacobi Eta, Theta and Zeta functions, while the parameter α is related to the energy eigenvalue E via the equation $E=\mathrm{dn}^2\alpha+k^2$, see Ref. [4]. They are also the eigenstates of the supercharge,

$$Q_1\Psi_E^{\pm}=\pm\sqrt{P_3(E)}\,\Psi_E^{\pm}.\qquad (12)$$

Assuming that the $j=2$ Lamé polynomials $\Psi_0^{(2)}=f+b$, $\Psi_1^{(2)}=\mathrm{cn}\,x\,\mathrm{dn}\,x$, $\Psi_2^{(2)}=\mathrm{sn}\,x\,\mathrm{dn}\,x$, $\Psi_3^{(2)}=\mathrm{sn}\,x\,\mathrm{cn}\,x$, $\Psi_4^{(2)}=f-b$ [9] form a zero-mode subspace of the fifth order integral Q_2, one finds

$$iQ_2=D^5+5fD^3+\tfrac{15}{2}f'D^2+\left(\tfrac{9}{2}f''+4f^2\right)D.\ (13)$$

In the same way for $j=3$ and $j=4$ a tedious calculation gives the supercharges Q_3 and Q_4. We do not display their explicit form here, but, instead, describe the general structure of the supercharges corresponding to $j=0,1,2,3,4$. First, one notes that if the derivative is assigned the homogeneity degree $d_h(D)=1$, in accordance with Eq. (7) the function f can be assigned $d_h(f)=2$, and then $d_h(H_j)=2$ and $d_h(Q_j)=2j+1$. Every supercharge has the leading term D^{2j+1}, the next term is of the form fD^{2j-1}, and every subsequent term decreases the order of the derivative on the right in one unit. The supercharges corresponding to the even cases $j=0,2,4$ contain the last term to be proportional to D. With this structure of the supercharges for the first cases of $j=0,\ldots,4$ at hands, we can fix now the form of the supercharges in the generic case $j=n$. Let us present the Hamiltonian operator in terms of the function f,

$$H_j=-D^2-h_j\left(f(x)-f(0)\right),\quad h_j=\frac{1}{3}j(j+1),\ (14)$$

and look for the supercharge in the form

$$iQ_j=D^{2j+1}+\alpha_jfD^{2j-1}+\beta_jf'D^{2j-2}+(\gamma_jb^2+\\ +\delta_jf^2)D^{2j-3}+\lambda_jf'''D^{2j-4}+\ldots,\qquad (15)$$

where in coefficients associated to the factors D^l, $l\geq 0$, it is necessary to include all the independent structures of homogeneity degree $d_h=2j+1-l$ given in terms of f and its derivatives modulo identities (9). Requiring $[Q_j,H_j]=0$, we can fix the first coefficients,

$$\alpha_j=h_j\left(j+\tfrac{1}{2}\right),\ \beta_j=\alpha_j\left(j-\tfrac{1}{2}\right),\ \gamma_j=\tfrac{6}{5}\beta_j(j-1),\\ \delta_j=\tfrac{5}{36}\gamma_j(j-6),\quad \lambda_j=-\tfrac{5}{72}\gamma_j\left(j-\tfrac{3}{2}\right)(j-2).\ (16)$$

These coefficients allow us to find a recurrence relation for the supercharges. We note that the kernel K_{2n} of the supercharge Q_j with $j=2n$ is spanned by the $4n+1$ functions

$$\varphi_a\cdot(1,f,\ldots,f^{n-1}),\ a=1,\ldots,4;\quad f^n,\qquad (17)$$

$\varphi_1=\mathrm{sn}\,x\,\mathrm{cn}\,x$, $\varphi_2=\mathrm{sn}\,x\,\mathrm{dn}\,x$, $\varphi_3=\mathrm{cn}\,x\,\mathrm{dn}\,x$, $\varphi_4=1$, which are linear combinations of the Lamé polynomials of the degree $2n$. For $j=2n+1$, the kernel K_{2n+1} is formed by the $4n+3$ functions

$$\chi_a\cdot(1,f,\ldots,f^{n-1}),\ a=1,...,4;\ \ \chi_af^n,\ a=1,2,3,\ (18)$$

with $\chi_1 = \mathrm{dn}\,x$, $\chi_2 = \mathrm{cn}\,x$, $\chi_3 = \mathrm{sn}\,x$, $\chi_4 = \mathrm{sn}\,x\,\mathrm{cn}\,x\,\mathrm{dn}\,x$, where for $n=0$ the states proportional to χ_4 are absent. In comparison with the kernel K_{j-2} of the supercharge Q_{j-2}, the kernel K_j of the supercharge Q_j includes four additional states. These are $\varphi_a f^{n-1}$, $a=1,2,3$, and f^n for $j=2n$, and $\chi_a f^{n-1}$, $a=1,2,3$, and $\chi_4 f^{n-1}$ for $j=2n+1$. Therefore, there should exist a relation

$$Q_j = \Lambda_j Q_{j-2}, \tag{19}$$

where Λ_j is a fourth order differential operator of homogeneity degree $d_h(\Lambda)=4$ of the form (modulo the first identity from (9))

$$\Lambda_j = D^4 + \tilde{\alpha}_j f D^2 + \tilde{\beta}_j f' D + \tilde{\gamma}_j b^2 + \tilde{\tau}_j f''. \tag{20}$$

Using Eqs. (15), (16), one finds the numerical coefficients of the operator Λ_j

$$\tilde{\alpha}_j = 2j(j-1)+1\,, \quad \tilde{\beta}_j = \tfrac{1}{3}j(4j^2-7)+\tfrac{3}{2}\,,$$
$$\tilde{\gamma}_j = j^2(j-1)^2\,, \quad \tilde{\tau}_j = \tfrac{1}{6}(j+3)(j+1)(j-1)^2. \tag{21}$$

The operator Λ_j is not symmetric, and in the representation (19) it serves to annihilate the additional zero modes of Q_j after application to them of the operator Q_{j-2}. There is also an alternative recurrence representation of the supercharge, $Q_j = Q_{j-2}\Lambda_j^\dagger$. The Hermitian conjugate operator $\Lambda_j^\dagger$ acts invariantly on the kernel of Q_{j-2}, $\Lambda_j^\dagger : K_{j-2} \to K_{j-2}$, and transforms four additional states of the kernel of Q_j into some linear combinations of the states of K_{j-2}. Relation (19) (or, $Q_j = Q_{j-2}\Lambda_j^\dagger$) allows ones to calculate Q_j for arbitrary even and odd values of j proceeding from the explicitly displayed supercharges Q_2 (or, Q_0) and Q_1.

With the fixed form of the integral Q_j, let us discuss the general structure of a hidden polynomial supersymmetry. Since Q_j is a self-conjugate odd local differential operator, one can introduce another, nonlocal supercharge, $\tilde{Q}_j = iRQ_j$. H_j and R are commuting self-conjugate operators. Let $\Psi^\pm_E$ be their common eigenstates, $H_j\Psi^\pm_E = E\Psi^\pm_E$, $R\Psi^\pm_E = \pm\Psi^\pm_E$. Since Q_j commutes with H_j and anticommutes with R, there exist some linearly independent combinations $\Psi_{E,q}$ and $R\Psi_{E,q}$ of Ψ^+_E and Ψ^-_E such that

$$Q_j\Psi_{E,q} = q(E)\Psi_{E,q}, \quad Q_jR\Psi_{E,q} = -q(E)R\Psi_{E,q}. \tag{22}$$

Then, the states $\Psi_{E,q}$ and $R\Psi_{E,q}$ are the eigenstates of Q_j^2 and $\tilde{Q}_j^2$ with the same eigenvalue $q^2(E)$, and, hence, the same is valid for the states $\Psi^\pm_E$, $Q_j^2\Psi^\pm_E = \tilde{Q}_j^2\Psi^\pm_E = q^2(E)\Psi^\pm_E$. All the states $\Psi^\pm_E$ corresponding to the allowed zones constitute the basis in the class of Bloch functions (including the periodic and antiperiodic ones). Hence, we get the operator equality $Q_j^2 = \tilde{Q}_j^2 = q^2(H_j)$. The operator Q_j^2 is a local differential operator of degree $4j+2$. This means that the operator $q^2(H_j)$ is a polynomial of degree $2j+1$ of its argument, i.e. $q^2(H_j) = C(H_j - c_0)(H_j-c_1)\ldots(H_j-c_{2j})$, where C is a real constant, while c_i are real, or some pairs of them could be mutually conjugate complex numbers. Comparing the coefficients in Q_j^2 and $q^2(H_j)$ before the operator D^{4j+2}, we find that $C=1$. Then, applying the operator $q^2(H_j)$ to the $2j+1$ (anti)periodic eigenstates of the Hamiltonian H_j corresponding to the boundaries of allowed zones $E_i(n)$, and remembering that the same states constitute the kernel K_j of the supercharge Q_j, we find that the set of the constants c_i coincides with the set of the boundary eigenvalues $E_i(n)$, $i=0,\ldots,2n$. Thus, we have shown that $Q_n^2 = \tilde{Q}_n^2 = P_{2n+1}(H_n)$, where

$$P_{2n+1}(E) := \textstyle\prod_{i=0}^{2n}(E - E_i(n)) \tag{23}$$

is the Lamé spectral polynomial. The nontrivial odd integrals generate the order $2n+1$ *polynomial superalgebra* being the hidden symmetry of the *bosonic* system (1).

The nonlinear character of the local supercharges and supersymmetry of the system (1) is reminiscent to a nonlinear symmetry of a particle in a Coulomb potential generated by the Laplace-Runge-Lenz vector integral, and to that of an anisotropic oscillator with commensurable frequencies [27]. Let us clarify the dynamical picture underlying the hidden nonlinear supersymmetry having in mind the analogy with the anisotropic oscillator. Consider the one-gap case. The Hamiltonian H_1 can be factorized in three possible ways:

$$H_1 = A_d^\dagger A_d + k^2 = A_c^\dagger A_c + 1 = A_s^\dagger A_s + 1 + k^2, \tag{24}$$

where $A_d = D - (\ln \mathrm{dn}\,x)'$, $A_d\,\mathrm{dn}\,x = 0$, and A_c and A_s have a similar structure in terms of the $\mathrm{cn}\,x$ and $\mathrm{sn}\,x$. Write the Heisenberg equations of motion of A_d and $A_s^\dagger$,

$$i\dot{A}_d = \omega_d(x)A_d, \quad i\dot{A}_s^\dagger = -A_s^\dagger\,\omega_s(x), \tag{25}$$

$\omega_d(x) = -2(\ln \mathrm{dn}\,x)''$, $\omega_s(x) = -2(\ln \mathrm{sn}\,x)''$. Define the operator $A_{s/d} = D - (\ln \mathrm{sn}\,x)' + (\ln \mathrm{dn}\,x)'$, for which

$$i\dot{A}_{s/d} = \omega_s(x)A_{s/d} - A_{s/d}\omega_d(x). \tag{26}$$

Then the relation

$$iA_s^\dagger A_{s/d}A_d = Q_1 \tag{27}$$

gives us one of the six possible factorizations of the supercharge (5). Note that operators A_c, A_s, $A_{s/d}$ and associated instant frequencies have singularities on a real line, which cancel in the H_1 and Q_1. For $j=n>1$ the same dynamical mechanism underlies the supercharge structure and its possible factorizations.

We conclude that the physical systems associated with the n-gap Lamé equation possess a hidden bosonized nonlinear supersymmetry. It is behind the double degeneration of the energy levels in the interior of the allowed bands and the singlet character of the $2n+1$ edge-bands states. The latter form a zero-mode subspace of the local supercharge Q_j (as well as of the nonlocal one, $\tilde{Q}_j$) being a differential operator of degree $2n+1$. Taking into account the parity of the states (17) and (18), one finds that the system (1) with any $j = n$ is characterized by the Witten index [28] equal to 1. The information on the transfer matrix can also be extracted from the structure of its hidden supersymmetry. The detailed analysis of this aspect will be presented in a separate publication.

In the limit $k = 1$, $\mathrm{sn}\, x = \tanh x$, $\mathrm{cn}\, x = \mathrm{dn}\, x = \mathrm{sech}\, x$, the valence bands $[E_0, E_1], \ldots, [E_{2n-2}, E_{2n-1}]$ shrink, and two boundary states of a valence band transform into one bound state of the related Pöschl-Teller system. As a result, the kernel of the supercharges of the latter system is constituted not only by the bound eigenstates and the lowest eigenstate from the continuous part of its spectrum, but also should include some n unbounded states. The discussion of this limit of the Lamé equation corresponding to the Pöschl-Teller problem, and its relation to the bound state Aharonov-Bohm and the Dirac delta potential systems from the viewpoint of the hidden supersymmetry will be presented elsewhere.

A simple shift of the argument in the Lamé equation for a half of the real period of the Hamiltonian H_j, $x \to x + K$, gives the isospectral doubly-periodic system

$$\tilde{H}_j = -D^2 + j(j+1)(1 - k'^2\, \mathrm{dn}^{-2}(x,k)). \qquad (28)$$

At $j = n$ it has a hidden polynomial supersymmetry generated by the supercharges Q_j and $\tilde{Q}_j$, whose explicit structure can be obtained by applying the Jacobi functions identities $\mathrm{sn}\,(x+K) = \mathrm{cn}\, x/\mathrm{dn}\, x$, $\mathrm{cn}\,(x+K) = -k' \,\mathrm{sn}\, x/\mathrm{dn}\, x$, $\mathrm{dn}\,(x+K) = k'/\mathrm{dn}\, x$ to the supercharges of the system (1). It would be interesting to look for the hidden polynomial supersymmetry in other finite-zone double periodic quantum systems.

Finally, it would be interesting to clarify the role played by the revealed hidden nonlinear supersymmetry of the finite-gap Lamé equation in the periodic Korteweg-de Vries equation theory [11] and in the periodic relativistic field theories [14, 15, 16] to which system (1) is intimately related.

The work has been supported partially by the FONDECYT Project 1050001 (MP), CONICYT PhD Program Fellowship (FC), Spanish Ministerio de Educación y Ciencia (Project MTM2005-09183) and Junta de Castilla y León (Excellence Project VA013C05) (LMN). LMN also thanks the Mecesup Project USA0108 for making possible his visit to the University of Santiago de Chile, and Department of Physics of this University for hospitality.

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